Estimation Problems

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Problem 1 (Heavy Estimates). Estimate the weight of the following living beings in kg:

- an ant,
- a goat,
- an elephant,
- a blue whale,
- a bacterium.

Problem 2 (Big Money). How many €50 banknotes would be needed to cover the entire surface of the Earth?

Problem 3 (Archimedes). Estimate the magnitude, in N, of the buoyant force currently acting on your body.

Problem 4 (Mass of the Atmosphere). What is the total mass of Earth's atmosphere, in kg?

Problem 5 (Mass of the Earth). Given that the average distance between the Earth and the Moon is approximately $3.8 \times 10^5\,\text{km}$, estimate the Earth's mass.

Problem 6 (Earth's Rotation). Using the previous estimate of Earth's mass, calculate its rotational kinetic energy about the polar axis.

Problem 7 (Mass of the Sun). Sunlight takes approximately $8.33\,\text{min}$ to reach the Earth. Based on this, estimate the number of protons in the Sun.

Problem 8 (Last Breath). Roughly how many air molecules enter your lungs during a single breath?

Problem 9 (**The Kingfisher** — from the Physics Team Competition)**.** A fisherman leans over the side of its boat, fills a glass with seawater (molar mass $18\,\text{g}\,\text{mol}^{-1}$), and pours it back into the ocean. Many years later, once the water has completely mixed with the oceans (which cover about 70% of the Earth's surface with an average depth of 4 km), the kingfisher returns to the same spot and fills the same glass again. How many molecules did the fishermen collect both times?

Problem 10 (**Tree Growth Time**)**.** The solar constant $C_S \sim 1\,\text{kW/m}^2$ describes the amount of solar radiation received per unit area. Through photosynthesis, plants convert a portion of this energy into biomass, with an efficiency of roughly 1%. Assuming the energy per unit mass stored in wood is approximately its calorific value, $P_C \sim 10^7\,\text{J/kg}$, estimate how long it takes for a tree to grow to 10 meters in height.

Solutions

Solution 1 (**Heavy Estimates**)**.** There are two ways to approach this: you can either guess based on experience or estimate the creature's volume and multiply it by the density of water. For something as small as a bacterium, the second method works best.

- ant $\approx 10^{-3}\,\text{kg}$,
- goat $\approx 10\,\text{kg}$,
- elephant $\approx 10^3\,\text{kg}$,
- blue whale $\approx 10^5\,\text{kg}$,
- bacterium $\approx 10^{-15} - 10^{-16}\,\text{kg}$.

Solution 2 (**Big Money**)**.** By comparing the total surface area of the Earth to the area of a single banknote, we estimate:

$$N \approx 5 \cdot 10^{16}.$$

Solution 3 (**Archimedes**)**.** Since the human body has a density close to that of water, a person weighing 50 kg displaces approximately 50 L of water. The resulting buoyant force is:

$$F_A = \rho_w V g \approx 50 \times 10^{-3}\,\text{m}^3 \cdot 10\,\text{m/s}^2 \approx 0.5\,\text{N}.$$

Solution 4 (**Mass of the Atmosphere**)**.** Assuming that the atmosphere's thickness is small compared to the Earth's radius, the gravitational field can be approximated as homogeneous. Hence

$$M_{\text{atm}} \approx \frac{4\pi R_T^2 P_0}{g} \approx 5 \times 10^{18}\,\text{kg}.$$

Solution 5 (**Mass of the Earth**)**.** Using Kepler's third law:

$$T^2 = \frac{4\pi^2 R^3}{G M_T},$$

we rearrange to find:

$$M_T = \frac{4\pi^2 R^3}{GT^2}.$$

Assuming that the orbital period of the Moon is about one month, this gives:

$$M_T \approx 6 \times 10^{24}\,\text{kg}.$$

Solution 6 (**Earth's Rotation**). With known values for Earth's mass and radius, we can approximate its moment of inertia as a uniform sphere. The rotational kinetic energy is then:

$$K_{\text{rot}} = \frac{1}{2} I\omega^2 \approx 10^{29}\,\text{J},$$

using the fact that Earth completes one rotation per day.

Solution 7 (**Mass of the Sun**). We use Kepler's law again, now with the distance derived from the 8.33-minute light travel time. After calculating the Sun's mass, we divide by the mass of a single proton. This yields approximately 10^{57} protons.

Solution 8 (**Last Breath**). Assuming an inhalation volume of about 1 liter, we use the ideal gas law to estimate:

$$N = \frac{P_{\text{atm}} V}{k_B T_{\text{room}}} \approx 2 \times 10^{22}.$$

Solution 9 (**The Kingfisher**). Set up the proportion:

$$x : n_{\text{glass}} = n_{\text{glass}} : N_{\text{ocean}} \quad \Longrightarrow \quad x = \frac{n_{\text{glass}}^2}{N_{\text{ocean}}}.$$

We calculate the ocean volume as a shell of depth $h = 4\,\text{km}$ over a sphere of radius R_T:

$$V_{\text{ocean}} = f \cdot 4\pi R_T^2 h,$$

with $f = 0.7$ (the fraction of Earth's surface covered by oceans). Let M_{a} be the molar mass of water and N_A Avogadro's number. Then:

$$N_{\text{ocean}} = V_{\text{ocean}} \cdot \frac{N_A \rho_{\text{a}}}{M_{\text{a}}}, \quad n_{\text{glass}} = V_{\text{glass}} \cdot \frac{N_A \rho_{\text{a}}}{M_{\text{a}}}.$$

Substituting:

$$x = \frac{n_{\text{glass}}^2}{N_{\text{ocean}}} = \frac{V_{\text{glass}}^2 \cdot \rho_{\text{a}} N_A}{4\pi R_T^2 h f M_{\text{a}}} \approx 10^3.$$

Solution 10 (**Tree Growth Time**). The solar energy received by a tree's leaves is partly converted into biomass. With 1% efficiency, the energy stored in new wood mass over a time Δt is:

$$\Delta E = C_s A \cdot \Delta t,$$

where A is the crown area. If the tree grows by Δh meters and its trunk radius is R, the added biomass stores:

$$P_C \cdot \Delta m = P_C \cdot \pi R^2 \Delta h \cdot \rho_{\text{wood}}.$$

Since only 1% of the incoming energy goes into growth:

$$\Delta t = \frac{100 \cdot P_C \cdot \pi R^2 \Delta h \cdot \rho_{\text{wood}}}{C_s A}.$$

We multiply by 2 to account for nighttime. With reasonable values, the growth time comes out to several decades.